

LIMITING BEHAVIOUR OF INTRINSIC SEMI-NORMS IN FRACTIONAL ORDER SOBOLEV SPACES

DEDICATED TO THE MEMORY OF CHARLES GOULAOUIC
(1938–1983)

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ABSTRACT. We collect and extend results on the limit of $\sigma^{1-k}(1 - \sigma)^k |v|_{l+\sigma,p,\Omega}^p$ as $\sigma \rightarrow 0^+$ or $\sigma \rightarrow 1^-$, where Ω is \mathbb{R}^n or a smooth bounded domain, $k \in \{0, 1\}$, $l \in \mathbb{N}$, $p \in [1, \infty)$, and $|\cdot|_{l+\sigma,p,\Omega}$ is the intrinsic semi-norm of order $l + \sigma$ in the Sobolev space $W^{l+\sigma,p}(\Omega)$. In general, the above limit is equal to $c[v]^p$, where c and $[\cdot]$ are, respectively, a constant and a semi-norm that we explicitly provide. The particular case $p = 2$ for $\Omega = \mathbb{R}^n$ is also examined and the results are then proved by using the Fourier transform.

1. INTRODUCTION

Bourgain, Brézis and Mironescu (cf. [5, 6]) proved that, for any $p \in [1, \infty)$ and any v belonging to the Sobolev space $W^{1,p}(\Omega)$,

$$(1.1) \quad \lim_{\sigma \rightarrow 1^-} (1 - \sigma) |v|_{\sigma,p,\Omega}^p = p^{-1} K_{p,n} \int_{\Omega} |\nabla v(x)|^p dx,$$

where Ω is either \mathbb{R}^n or a smooth bounded domain in \mathbb{R}^n , with $n \geq 1$, $|\cdot|_{\sigma,p,\Omega}$ is the intrinsic or Gagliardo semi-norm of order σ in the Sobolev space $W^{\sigma,p}(\Omega)$ (see Section 2 for the precise definitions), and $K_{p,n}$ is a constant that only depends on p and n . Likewise, Maz'ya and Shaposhnikova [12] showed that

$$(1.2) \quad \lim_{\sigma \rightarrow 0^+} \sigma |v|_{\sigma,p,\mathbb{R}^n}^p = 2p^{-1} |S_{n-1}| |v|_{0,p,\mathbb{R}^n}^p,$$

where S_{n-1} stands for the unit sphere in \mathbb{R}^n (i.e. $S_{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$) and $|S_{n-1}|$ is its area.

These results have been extended and completed by several authors. Let us quote, for example, Milman [13], who placed them in the frame of interpolation spaces, or Karadzhov, Milman and Xiao [9], Kolyada and Lerner [10] and Triebel [14], who generalized them in the context of Besov spaces.

Our interest in this subject comes from the study of sampling inequalities involving Sobolev semi-norms. In [4], we have extended previous results

Date: December 1, 2011.

2010 Mathematics Subject Classification. Primary 46E35; Secondary 46E30, 46F12.

Key words and phrases. Sobolev spaces, fractional order semi-norms, Fourier transform, Beppo-Levi spaces.

(cf. [2, 3]) in order to allow fractional order Sobolev semi-norms on the left-hand side of sampling inequalities. We have then realized that the complete comprehension of the constants involved in sampling inequalities needs an understanding of the asymptotic behaviour of the corresponding fractional order Sobolev semi-norms. In fact, we need extensions of (1.1) and (1.2) having the following form:

$$(1.3) \quad \lim_{\sigma \rightarrow \ell} \sigma^{1-k} (1 - \sigma)^k |v|_{l+\sigma, p, \Omega}^p = c[v]^p,$$

where $\ell = 0^+$ or 1^- , Ω is \mathbb{R}^n or a smooth bounded domain, $k \in \{0, 1\}$, $l \in \mathbb{N}$, $p \in [1, \infty)$, and $|\cdot|_{l+\sigma, p, \Omega}$ is the intrinsic semi-norm of order $l + \sigma$ in the Sobolev space $W^{l+\sigma, p}(\Omega)$. On the right-hand side of (1.3), the notations $[\cdot]$ and c stand, respectively, for a semi-norm and a constant to be specified.

The first part of this paper will be devoted to establish (1.3). Most of the work may be routine, but anyway we find it useful to collect and state in one place this kind of results and to provide explicit expressions of the constants and semi-norms involved in the limits.

In the second part of the paper, we shall focus on the case $p = 2$ and $\Omega = \mathbb{R}^n$. We show that (1.3) can be obtained by means of the Fourier transform. This line of reasoning was suggested in [5, Remark 2] starting from a result by Masja and Nagel [11]. As a by-product, for $m \in \mathbb{N}$ and $s \geq 0$, we establish a relationship between the Sobolev space $W^{m+s, 2}(\mathbb{R}^n)$ and the Beppo-Levi space $X^{m, s}$, which is a space that arises in spline theory (cf. [1, Chapter I]).

2. PRELIMINARIES

For any $x \in \mathbb{R}$, we shall write $\lfloor x \rfloor$ for the *floor* (or integer part) of x , that is, the unique integer satisfying $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. The letter n will always stand for an integer belonging to $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ (by convention, $0 \in \mathbb{N}$). The Euclidean norm in \mathbb{R}^n will be denoted by $|\cdot|$.

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial^\alpha = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$, x_1, \dots, x_n being the generic independent variables in \mathbb{R}^n . In addition, given $l \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $\binom{l}{\alpha} = l! / (\alpha_1! \dots \alpha_n!)$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. We shall make frequent use of the relation

$$(2.1) \quad |x|^{2l} = \sum_{|\alpha|=l} \binom{l}{\alpha} x^{2\alpha},$$

valid for any $l \in \mathbb{N}$ and $x \in \mathbb{R}^n$.

Let Ω be a nonempty open set in \mathbb{R}^n . For any $r \in \mathbb{N}$ and for any $p \in [1, \infty)$, we shall denote by $W^{r, p}(\Omega)$ the usual Sobolev space defined by

$$W^{r, p}(\Omega) = \{ v \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n, |\alpha| \leq r, \partial^\alpha v \in L^p(\Omega) \}.$$

We recall that the derivatives $\partial^\alpha v$ are taken in the distributional sense. The space $W^{r, p}(\Omega)$ is equipped with the semi-norms $|\cdot|_{j, p, \Omega}$, with $j \in \{0, \dots, r\}$,

and the norm $\|\cdot\|_{r,p,\Omega}$ given by

$$|v|_{j,p,\Omega} = \left(\sum_{|\alpha|=j} \int_{\Omega} |\partial^{\alpha} v(x)|^p dx \right)^{1/p} \quad \text{and} \quad \|v\|_{r,p,\Omega} = \left(\sum_{j=0}^r |v|_{j,p,\Omega}^p \right)^{1/p}.$$

For any $r \in (0, \infty) \setminus \mathbb{N}$ and for any $p \in [1, \infty)$, we shall denote by $W^{r,p}(\Omega)$ the Sobolev space of noninteger order r , formed by the (equivalence classes of) functions $v \in W^{\lfloor r \rfloor, p}(\Omega)$ such that

$$|v|_{r,p,\Omega}^p = \sum_{|\alpha|=\lfloor r \rfloor} \int_{\Omega \times \Omega} \frac{|\partial^{\alpha} v(x) - \partial^{\alpha} v(y)|^p}{|x - y|^{n+p(r-\lfloor r \rfloor)}} dx dy < \infty.$$

Besides the semi-norms $|\cdot|_{j,p,\Omega}$, with $j \in \{0, \dots, \lfloor r \rfloor\}$, and $|\cdot|_{r,p,\Omega}$, the space $W^{r,p}(\Omega)$ is endowed with the norm

$$\|v\|_{r,p,\Omega} = \left(\|v\|_{\lfloor r \rfloor, p, \Omega}^p + |v|_{r,p,\Omega}^p \right)^{1/p}.$$

Given $j \in \mathbb{N}$ and $v \in W^{j+1,p}(\Omega)$, we put

$$|\nabla v|_{0,p,\Omega} = \left(\int_{\Omega} |\nabla v(x)|^p dx \right)^{1/p} \quad \text{and} \quad |\nabla v|_{j,p,\Omega} = \left(\sum_{|\alpha|=j} |\nabla(\partial^{\alpha} v)|_{0,p,\Omega}^p \right)^{1/p}.$$

The mapping $v \mapsto |\nabla v|_{j,p,\Omega}$ is a semi-norm in $W^{j+1,p}(\Omega)$ equivalent to $|\cdot|_{j+1,p,\Omega}$.

We shall use the following definition of the Fourier transform \hat{v} of a function $v \in L^1(\mathbb{R}^n)$:

$$\forall \xi \in \mathbb{R}^n, \quad \hat{v}(\xi) = \int_{\mathbb{R}^n} v(x) e^{-ix \cdot \xi} dx.$$

We refer to standard textbooks for the properties of the Fourier transform and their extension to the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. We just recall the following result:

$$(2.2) \quad \forall v \in \mathcal{S}'(\mathbb{R}^n), \forall \alpha \in \mathbb{N}^n, \quad i^{|\alpha|} \xi^{\alpha} \hat{v} = \widehat{\partial^{\alpha} v}.$$

3. GENERAL RESULTS FOR $p \in [1, \infty)$

As mentioned in the introduction, for a smooth bounded domain Ω or for $\Omega = \mathbb{R}^n$, we are interested in obtaining the following limit:

$$(3.1) \quad \lim_{\sigma \rightarrow \ell} \sigma^{1-k} (1 - \sigma)^k |v|_{l+\sigma,p,\Omega}^p,$$

with $\ell \in \{0^+, 1^-\}$, $k \in \{0, 1\}$, $l \in \mathbb{N}$, $p \in [1, \infty)$ and v belonging to a suitable Sobolev space. For $\Omega = \mathbb{R}^n$, we shall study the cases $(\ell, k) = (0^+, 0)$ and $(1^-, 1)$, whereas, for Ω bounded, we shall consider the cases $(\ell, k) = (0^+, 1)$ and $(1^-, 1)$, taking into account that $\lim_{\sigma \rightarrow 0^+} (1 - \sigma) = 1$. The limit corresponding to any other combination of ℓ and k follows trivially from the above cases.

Theorem 3.1. *Let Ω be a bounded domain in \mathbb{R}^n with a Lipschitz-continuous boundary. Let $p \in [1, \infty)$ and $l \in \mathbb{N}$. Then, for any $v \in W^{l+1,p}(\Omega)$,*

$$(3.2) \quad \lim_{\sigma \rightarrow 1^-} (1 - \sigma) |v|_{l+\sigma,p,\Omega}^p = p^{-1} K_{p,n} |\nabla v|_{l,p,\Omega}^p,$$

where

$$(3.3) \quad K_{p,n} = \int_{S_{n-1}} |\omega \cdot \nu|^p d\omega,$$

ν being any unit vector in \mathbb{R}^n .

Proof. The case $l = 0$ is a result by Bourgain, Brézis and Mironescu (cf. [5]). For the sake of completeness, we just clarify here some details of their proof. We use, however, the notations in [6], which are slightly simpler. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be any family of nonnegative functions, contained in $L^1_{\text{loc}}(0, \infty)$, such that

$$\int_0^\infty \rho_\varepsilon(t) t^{n-1} dt = 1, \quad \forall \varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_\delta^\infty \rho_\varepsilon(t) t^{n-1} dt = 0, \quad \forall \delta > 0.$$

It follows from Theorems 2 and 3 in [5] that, for any $v \in W^{1,p}(\Omega)$,

$$(3.4) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx dy = K_{p,n} |\nabla v|_{0,p,\Omega}^p,$$

where $K_{p,n}$ is defined by (3.3). Let us choose the family $(\rho_\varepsilon)_{\varepsilon>0}$ given by

$$\rho_\varepsilon(t) = \begin{cases} \varepsilon d^{-\varepsilon} t^{\varepsilon-n}, & \text{if } t \leq d, \\ 0, & \text{if } t > d, \end{cases}$$

d being the diameter of Ω . Then, (3.4) becomes

$$\lim_{\varepsilon \rightarrow 0} \varepsilon d^{-\varepsilon} \int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^p}{|x - y|^{n+p-\varepsilon}} dx dy = K_{p,n} |\nabla v|_{0,p,\Omega}^p,$$

which implies (3.2), for $l = 0$, if we replace ε by $p(1 - \sigma)$.

Let us now consider the case $l \geq 1$. Since the l th-order derivatives of functions in $W^{l+1,p}(\Omega)$ belong to $W^{1,p}(\Omega)$, by the case $l = 0$, for any $v \in W^{l+1,p}(\Omega)$, we have

$$\begin{aligned} \lim_{\sigma \rightarrow 1^-} (1 - \sigma) |v|_{l+\sigma,p,\Omega}^p &= \lim_{\sigma \rightarrow 1^-} (1 - \sigma) \sum_{|\alpha|=l} |\partial^\alpha v|_{\sigma,p,\Omega}^p \\ &= \sum_{|\alpha|=l} \lim_{\sigma \rightarrow 1^-} (1 - \sigma) |\partial^\alpha v|_{\sigma,p,\Omega}^p \\ &= \sum_{|\alpha|=l} p^{-1} K_{p,n} |\nabla(\partial^\alpha v)|_{0,p,\Omega}^p = p^{-1} K_{p,n} |\nabla v|_{l,p,\Omega}^p, \end{aligned}$$

which yields (3.2). □

Remark 3.2. Let us provide the explicit value of the constant $K_{p,n}$ given by (3.3). Since the definition of $K_{p,n}$ is independent of the unit vector ν , we can take $\nu = (1, 0, \dots, 0)$. On the one hand, we have

$$\begin{aligned} \int_{x_1^2 + \dots + x_n^2 \leq 1} |x_1|^p dx &= \int_0^1 \left(\int_{S_{n-1}} t^{n-1} |t\omega_1|^p d\omega \right) dt \\ &= \left(\int_{S_{n-1}} |\omega \cdot \nu|^p d\omega \right) \int_0^1 t^{n-1+p} dt = \frac{K_{p,n}}{n+p}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{x_1^2 + \dots + x_n^2 \leq 1} |x_1|^p dx &= \int_{-1}^1 |x_1|^p \left(\int_{x_2^2 + \dots + x_n^2 \leq 1 - x_1^2} dx_2 \cdots dx_n \right) dx_1 \\ &= \vartheta_{n-1} \int_{-1}^1 |x_1|^p (1 - x_1^2)^{(n-1)/2} dx_1 = 2\vartheta_{n-1} \int_0^1 x_1^p (1 - x_1^2)^{(n-1)/2} dx_1 \\ &= \vartheta_{n-1} \int_0^1 t^{(p-1)/2} (1 - t)^{(n-1)/2} dt = \vartheta_{n-1} B\left(\frac{p+1}{2}, \frac{n+1}{2}\right), \end{aligned}$$

where ϑ_{n-1} is the volume of the unit ball in \mathbb{R}^{n-1} and B is the Euler Beta function. Hence,

$$(3.5) \quad K_{p,n} = (n+p)\vartheta_{n-1} B\left(\frac{p+1}{2}, \frac{n+1}{2}\right) = \frac{2\pi^{(n-1)/2} \Gamma((p+1)/2)}{\Gamma((n+p)/2)},$$

where Γ stands for the Euler Gamma function. Although Theorem 3.1 only requires the value of $K_{p,n}$ for $p \geq 1$, the above expression is valid, in fact, for any $p \geq 0$. \square

Theorem 3.3. *Let $p \in [1, \infty)$ and $l \in \mathbb{N}$. Then, for any $v \in W^{l+1,p}(\mathbb{R}^n)$,*

$$(3.6) \quad \lim_{\sigma \rightarrow 1^-} (1 - \sigma) |v|_{l+\sigma,p,\mathbb{R}^n}^p = p^{-1} K_{p,n} |\nabla v|_{l,p,\mathbb{R}^n}^p,$$

where $K_{p,n}$ is given by (3.3).

Proof. This result, for $l = 0$, is usually credited to Bourgain, Brézis and Mironescu [5], since it is implicitly contained in their paper. It can be proved from Theorem 3.1, first for smooth functions with compact support and then, by density, for any element in $W^{l+1,p}(\mathbb{R}^n)$. An explicit proof is given by Milman [13, Subsection 3.1], but without providing the precise definition of the constant $K_{p,n}$, which can be deduced from Karadzhov, Milman and Xiao [9, p. 332]. The case $l > 0$ is identical to that in the proof of Theorem 3.1. \square

Theorem 3.4. *Let $p \in [1, \infty)$, $l \in \mathbb{N}$ and $\sigma_0 \in (0, 1)$. Then, for any $v \in W^{l+\sigma_0,p}(\mathbb{R}^n)$,*

$$(3.7) \quad \lim_{\sigma \rightarrow 0^+} \sigma |v|_{l+\sigma,p,\mathbb{R}^n}^p = \frac{4\pi^{n/2}}{p \Gamma(n/2)} |v|_{l,p,\mathbb{R}^n}^p.$$

Proof. Maz'ya and Shaposhnikova proved in [12, Theorem 3] that (1.2) holds for any v belonging to $\bigcup_{0 < \sigma < 1} W_0^{\sigma,p}(\mathbb{R}^n)$, where $W_0^{\sigma,p}(\mathbb{R}^n)$ stands for the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to $|\cdot|_{\sigma,p,\mathbb{R}^n}$ (which is a norm in this last space). The condition on v can be relaxed to $v \in \bigcup_{0 < \sigma < \sigma_0} W_0^{\sigma,p}(\mathbb{R}^n)$ for some $\sigma_0 \in (0, 1)$. Likewise, since $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{\sigma,p}(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\sigma,p,\mathbb{R}^n} = \left(|\cdot|_{0,p,\mathbb{R}^n}^p + |\cdot|_{\sigma,p,\mathbb{R}^n}^p\right)^{1/p}$, it follows that $W^{\sigma,p}(\mathbb{R}^n) \subset W_0^{\sigma,p}(\mathbb{R}^n)$. Thus, taking into account the embedding $W^{\sigma_0,p}(\mathbb{R}^n) \hookrightarrow W^{\sigma,p}(\mathbb{R}^n)$, if $\sigma_0 \geq \sigma$, and that $|S_{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$, we conclude that, for $l = 0$, (3.7) follows from Maz'ya and Shaposhnikova's result.

Now, let us assume that $l \geq 1$. Given $v \in W^{l+\sigma_0,p}(\mathbb{R}^n)$, it is clear that any l th-derivative $\partial^\alpha v$ belongs to $W^{\sigma_0,p}(\mathbb{R}^n)$. The case $l = 0$ implies that

$$\begin{aligned} \lim_{\sigma \rightarrow 0^+} \sigma |v|_{l+\sigma,p,\mathbb{R}^n}^p &= \lim_{\sigma \rightarrow 0^+} \sigma \sum_{|\alpha|=l} |\partial^\alpha v|_{\sigma,p,\mathbb{R}^n}^p \\ &= \sum_{|\alpha|=l} \lim_{\sigma \rightarrow 0^+} \sigma |\partial^\alpha v|_{\sigma,p,\mathbb{R}^n}^p = \sum_{|\alpha|=l} \frac{4\pi^{n/2}}{p \Gamma(n/2)} |\partial^\alpha v|_{0,p,\mathbb{R}^n}^p = \frac{4\pi^{n/2}}{p \Gamma(n/2)} |v|_{l,p,\mathbb{R}^n}^p. \end{aligned}$$

The theorem follows. \square

As we shall next see, there exists a qualitative difference in the behaviour of $|v|_{l+\sigma,p,\Omega}$ as $\sigma \rightarrow 0^+$ depending on whether Ω is \mathbb{R}^n or a bounded set. Theorem 3.4 implies that the semi-norm $|v|_{l+\sigma,p,\mathbb{R}^n}$ blows up to infinity (except for polynomials of degree $\leq l$) as $\sigma \rightarrow 0^+$. However, for a bounded set Ω , a priori, the semi-norm $|v|_{l+\sigma,p,\Omega}$ may remain bounded. In fact, this is always the case. Even more, as $\sigma \rightarrow 0^+$, that semi-norm tends to Dini's semi-norm $|v|_{l,\text{Dini}(p),\Omega}$, defined, following Milman [13], by

$$|v|_{l,\text{Dini}(p),\Omega}^p = \sum_{|\alpha|=l} \int_{\Omega \times \Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^p}{|x - y|^n} dx dy.$$

Let us state and establish this result. We borrow the arguments from Milman [13, Theorem 3 and Example 2].

Theorem 3.5. *Let Ω be a bounded domain of \mathbb{R}^n with a Lipschitz continuous boundary. Let $p \in [1, \infty)$, $l \in \mathbb{N}$ and $\sigma_0 \in (0, 1)$. Then, for any $v \in W^{l+\sigma_0,p}(\Omega)$, we have $|v|_{l,\text{Dini}(p),\Omega} < \infty$ and*

$$\lim_{\sigma \rightarrow 0^+} |v|_{l+\sigma,p,\Omega} = |v|_{l,\text{Dini}(p),\Omega}.$$

Proof. As in previous results, it suffices to prove the case $l = 0$. Let R be the diameter of Ω . We consider the bijective linear mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $F(\hat{x}) = R\hat{x}$ and we write $\hat{\Omega} = F^{-1}(\Omega)$. Since $R = \text{diam } \Omega$, it is clear that $\text{diam } \hat{\Omega} = 1$. Thus,

$$\forall \sigma \in (0, \sigma_0), \forall \hat{x}, \hat{y} \in \hat{\Omega}, 1 \geq |\hat{x} - \hat{y}|^\sigma \geq |\hat{x} - \hat{y}|^{\sigma_0}.$$

Consequently, given $\hat{v} \in W^{\sigma_0,p}(\hat{\Omega})$, we have

$$\forall \sigma \in (0, \sigma_0), |\hat{v}|_{0,\text{Dini}(p),\hat{\Omega}}^p \leq |\hat{v}|_{\sigma,p,\hat{\Omega}}^p \leq |\hat{v}|_{\sigma_0,p,\hat{\Omega}}^p < \infty.$$

Hence, by the Lebesgue's Dominated Convergence Theorem, we get

$$\begin{aligned} \lim_{\sigma \rightarrow 0^+} |\hat{v}|_{\sigma,p,\hat{\Omega}}^p &= \lim_{\sigma \rightarrow 0^+} \int_{\hat{\Omega} \times \hat{\Omega}} \frac{|\hat{v}(\hat{x}) - \hat{v}(\hat{y})|^p}{|\hat{x} - \hat{y}|^{n+p\sigma}} d\hat{x} d\hat{y} \\ &= \int_{\hat{\Omega} \times \hat{\Omega}} \lim_{\sigma \rightarrow 0^+} \frac{|\hat{v}(\hat{x}) - \hat{v}(\hat{y})|^p}{|\hat{x} - \hat{y}|^{n+p\sigma}} d\hat{x} d\hat{y} \\ &= \int_{\hat{\Omega} \times \hat{\Omega}} \frac{|\hat{v}(\hat{x}) - \hat{v}(\hat{y})|^p}{|\hat{x} - \hat{y}|^n} d\hat{x} d\hat{y} = |\hat{v}|_{0,\text{Dini}(p),\hat{\Omega}}^p. \end{aligned}$$

Now, for any $v \in W^{\sigma_0,p}(\Omega)$, the function $\hat{v} = v \circ F$ belongs to $W^{\sigma_0,p}(\hat{\Omega})$, since

$$|v|_{\sigma_0,p,\Omega} = R^{-\sigma_0+n/p} |\hat{v}|_{\sigma_0,p,\hat{\Omega}}.$$

Likewise,

$$|v|_{0,\text{Dini}(p),\Omega} = R^{n/p} |\hat{v}|_{0,\text{Dini}(p),\hat{\Omega}}$$

and, for any $\sigma \in (0, \sigma_0)$,

$$|v|_{\sigma,p,\Omega} = R^{-\sigma+n/p} |\hat{v}|_{\sigma,p,\hat{\Omega}}.$$

From these relations, we deduce that $|v|_{0,\text{Dini}(p),\Omega}$ is finite and that

$$\lim_{\sigma \rightarrow 0^+} |v|_{\sigma,p,\Omega} = \lim_{\sigma \rightarrow 0^+} R^{-\sigma+n/p} |\hat{v}|_{\sigma,p,\hat{\Omega}} = R^{n/p} |\hat{v}|_{0,\text{Dini}(p),\hat{\Omega}} = |v|_{0,\text{Dini}(p),\Omega}.$$

The proof is complete. \square

Remark 3.6. It is worth noting that, under the conditions of Theorem 3.5, the arguments in its proof lead, in general, to the following bound:

$$\forall v \in W^{l+\sigma_0,p}(\Omega), \quad |v|_{l,\text{Dini}(p),\Omega} \leq R^\sigma |v|_{l+\sigma,p,\Omega} \leq R^{\sigma_0} |v|_{l+\sigma_0,p,\Omega},$$

with $R = \text{diam } \Omega$. \square

Remark 3.7. By a change of variables and Fubini's Theorem, it can be seen that

$$|v|_{0,\text{Dini}(p),\Omega} = \left(n \int_0^{+\infty} \frac{\bar{\omega}(v,t)_p^p}{t} dt \right)^{1/p},$$

where $\bar{\omega}(v,t)_p$ is the averaged modulus of smoothness, given by

$$\bar{\omega}(v,t)_p^p = t^{-n} \int_{|h| \leq t} |\Delta_h v|_{0,p,\Omega}^p dh, \quad t > 0,$$

with $\Delta_h v(x) = v(x+h) - v(x)$, if $x, x+h \in \Omega$, and $\Delta_h f(x) = 0$, otherwise. Hence, for $l = 0$, Theorem 3.5 establishes that, for any $v \in W^{\sigma_0,p}(\Omega)$, the function $\bar{\omega}(v, \cdot)_p$ satisfies a Dini-type condition. Analogous comments can be made for $l > 0$. This justifies the name given to the semi-norm $|\cdot|_{l,\text{Dini}(p),\Omega}$. Likewise, since $\bar{\omega}(v,t)_p$ is equivalent to the usual modulus of smoothness $\omega(v,t)_p = \sup_{|h| \leq t} |\Delta_h v|_{0,p,\Omega}$, Theorem 3.5 comprises as a particular case the result given by Milman (cf. [13, Example 2]). \square

Remark 3.8. The semi-norm $|\cdot|_{r,p,\mathbb{R}^n}$ can be normalized as follows:

$$(3.8) \quad [v]_{r,p,\mathbb{R}^n} = \lambda_{\sigma,p} |v|_{r,p,\mathbb{R}^n},$$

where $\sigma = r - \lfloor r \rfloor$ and

$$\lambda_{\sigma,p} = \begin{cases} (\sigma(1-\sigma))^{1/p}, & \text{if } \sigma \in (0, 1), \\ 1, & \text{if } \sigma = 0. \end{cases}$$

Then, the semi-norm $[\cdot]_{r,p,\mathbb{R}^n}$ is continuous in the scale of Sobolev spaces $(W^{r,p}(\mathbb{R}^n))_{r \geq 0}$ in the following sense:

$$\begin{aligned} \forall r > 0, \quad \forall v \in W^{r,p}(\mathbb{R}^n), \quad \lim_{s \rightarrow r^-} [v]_{s,p,\mathbb{R}^n} &\approx [v]_{r,p,\mathbb{R}^n}, \\ \forall r \geq 0, \quad \forall \epsilon > 0, \quad \forall v \in W^{r+\epsilon,p}(\mathbb{R}^n), \quad \lim_{s \rightarrow r^+} [v]_{s,p,\mathbb{R}^n} &\approx [v]_{r,p,\mathbb{R}^n}, \end{aligned}$$

where the symbol \approx means that there exist two positive constants c_1 and c_2 , independent of v , such that

$$c_1 [v]_{r,p,\mathbb{R}^n} \leq \lim_{s \rightarrow r^\pm} [v]_{s,p,\mathbb{R}^n} \leq c_2 [v]_{r,p,\mathbb{R}^n}.$$

In fact, if $r \notin \mathbb{N}$, both lateral limits are equal to $[v]_{r,p,\mathbb{R}^n}$. For $r \in \mathbb{N}$, these relations are direct consequences of Theorems 3.3 and 3.4, whereas, for $r \notin \mathbb{N}$, they come from the Lebesgue's Dominated Convergence Theorem.

For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz continuous boundary, we could also consider the normalization $[v]_{r,p,\Omega} = \lambda_{\sigma,p} |v|_{r,p,\Omega}$. But, due to Theorem 3.5, for any $r \in \mathbb{N}$, we would get

$$\forall \epsilon > 0, \forall v \in W^{r+\epsilon,p}(\Omega), \lim_{s \rightarrow r^+} [v]_{s,p,\Omega} = 0,$$

which is quite unnatural. A better normalization is

$$[v]_{r,p,\Omega} = (1 - \sigma)^{1/p} |v|_{r,p,\Omega},$$

with $\sigma = r - [r]$. We now have:

$$\begin{aligned} \forall r > 0, r \notin \mathbb{N}, \forall v \in W^{r,p}(\Omega), \lim_{s \rightarrow r^-} [v]_{s,p,\Omega} &\approx [v]_{r,p,\Omega}, \\ \forall r \geq 0, \forall \epsilon > 0, \forall v \in W^{r+\epsilon,p}(\Omega), \lim_{s \rightarrow r^+} [v]_{s,p,\Omega} &\approx \begin{cases} [v]_{r,p,\Omega}, & \text{if } r \notin \mathbb{N}, \\ |v|_{r,\text{Dini}(p),\Omega}, & \text{if } r \in \mathbb{N}. \end{cases} \end{aligned}$$

Observe that, given $r \in \mathbb{N}$ and $\varepsilon > 0$, the semi-norms $|\cdot|_{r,\text{Dini}(p),\Omega}$ and $|\cdot|_{r,p,\Omega}$ are not equivalent on $W^{r+\varepsilon,p}(\Omega)$ ($|\cdot|_{r,\text{Dini}(p),\Omega}$ is null for polynomials of degree $\leq r$, while $|\cdot|_{r,p,\Omega}$ is null only for polynomials of degree $\leq r - 1$). Consequently, the semi-norm $[\cdot]_{r,p,\Omega}$ is not right-continuous for $r \in \mathbb{N}$. \square

4. THE PARTICULAR CASE $p = 2$

The purpose of this section is to provide an alternative proof of Theorems 3.3 and 3.4 based on the Fourier transform. We start with several preliminary results.

Lemma 4.1. *For any $n \in \mathbb{N}$ and $\sigma \in (0, 1)$, there exists a positive constant $G_{\sigma,n}$ such that*

$$(4.1) \quad \forall \xi \in \mathbb{R}^n, \int_{\mathbb{R}^n} \frac{|e^{i\xi \cdot y} - 1|^2}{|y|^{n+2\sigma}} dy = G_{\sigma,n} |\xi|^{2\sigma}.$$

Proof. The relation (4.1) is obviously true if $\xi = 0$, so let us assume that $\xi \neq 0$. Let $\nu = \xi/|\xi|$. We have

$$\begin{aligned}
& \int_{\mathbb{R}^n} \frac{|e^{i\xi \cdot y} - 1|^2}{|y|^{n+2\sigma}} dy \\
&= |\xi|^{2\sigma} \int_{\mathbb{R}^n} \frac{|e^{i\nu \cdot x} - 1|^2}{|x|^{n+2\sigma}} dx \quad (\text{change } x = |\xi|y) \\
&= |\xi|^{2\sigma} \int_{S_{n-1}} \left(\int_0^{+\infty} \rho^{n-1} \frac{|e^{i\rho\nu \cdot \omega} - 1|^2}{|\rho\omega|^{n+2\sigma}} d\rho \right) d\omega \\
&= |\xi|^{2\sigma} \int_{S_{n-1}} \left(\int_0^{+\infty} \frac{|e^{i\rho\nu \cdot \omega} - 1|^2}{\rho^{1+2\sigma}} d\rho \right) d\omega \\
&= |\xi|^{2\sigma} \int_{S_{n-1}} \left(\int_0^{+\infty} \frac{2(1 - \cos(\rho\nu \cdot \omega))}{\rho^{1+2\sigma}} d\rho \right) d\omega \\
&= |\xi|^{2\sigma} \int_{S_{n-1}} \left(\int_0^{+\infty} \frac{2(1 - \cos(\rho|\nu \cdot \omega|))}{\rho^{1+2\sigma}} d\rho \right) d\omega \quad (\cos \text{ is even}) \\
&= |\xi|^{2\sigma} \int_{S_{n-1}} |\nu \cdot \omega|^{2\sigma} \left(\int_0^{+\infty} \frac{2(1 - \cos t)}{t^{1+2\sigma}} dt \right) d\omega \quad (\text{change } t = \rho|\nu \cdot \omega|) \\
&= |\xi|^{2\sigma} K_{2\sigma,n} M_\sigma,
\end{aligned}$$

where $K_{2\sigma,n}$ is given by (3.3) with $p = 2\sigma$ and

$$(4.2) \quad M_\sigma = \int_0^{+\infty} \frac{2(1 - \cos t)}{t^{1+2\sigma}} dt,$$

which is convergent for any $\sigma \in (0, 1)$. It then suffices to take $G_{\sigma,n} = K_{2\sigma,n} M_\sigma$. \square

Remark 4.2. For any $n \in \mathbb{N}$ and $\sigma \in (0, 1)$, let us show that

$$(4.3) \quad G_{\sigma,n} = \frac{2\pi^{(n+1)/2} \Gamma(\sigma + 1/2)}{\Gamma(\sigma + n/2) \Gamma(1 + 2\sigma) \sin(\pi\sigma)}.$$

Integrating by parts in (4.2), we get

$$M_\sigma = -\left. \frac{(1 - \cos t)}{\sigma t^{2\sigma}} \right|_0^{+\infty} + \int_0^{+\infty} \frac{\sin t}{\sigma t^{2\sigma}} dt = \frac{1}{\sigma} \int_0^{+\infty} \frac{\sin t}{t^{2\sigma}} dt.$$

This last integral can be computed in several ways. For example, the cases $\sigma \in (0, 1/2)$, $\sigma = 1$ and $\sigma \in (1/2, 1)$ are covered, respectively, by relations 3.764.1, 3.741.2 and, after an integration by parts, 3.764.2 in Gradshteyn and Ryzhik [8]. Using well-known properties of the Gamma function, as well as the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z},$$

we finally derive that

$$(4.4) \quad M_\sigma = \frac{\pi}{\Gamma(1 + 2\sigma) \sin(\pi\sigma)},$$

Since $G_{\sigma,n} = K_{2\sigma,n} M_\sigma$, this relation, together with (3.5), implies (4.3). \square

Proposition 4.3 (C. Goulaouic). *Let $\sigma \in (0, 1)$. Then*

$$(4.5) \quad W^{\sigma,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap \widetilde{H}^\sigma(\mathbb{R}^n),$$

with

$$(4.6) \quad \widetilde{H}^\sigma = \left\{ v \in \mathcal{S}'(\mathbb{R}^n) \mid \hat{v} \in L^1_{\text{loc}}(\mathbb{R}^n), \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi < \infty \right\}.$$

In fact, for any $v \in W^{\sigma,2}(\mathbb{R}^n)$,

$$(4.7) \quad |v|_{\sigma,2,\mathbb{R}^n}^2 = (2\pi)^{-n} G_{\sigma,n} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi,$$

where $G_{\sigma,n}$ is the constant given by Lemma 4.1.

Proof. Let $v \in L^2(\mathbb{R}^n)$. We first remark that v is, in particular, a tempered distribution and, by Plancherel's Theorem, $\hat{v} \in L^2(\mathbb{R}^n)$, so $\hat{v} \in L^1_{\text{loc}}(\mathbb{R}^n)$. Thus, to prove (4.5), it suffices to see that the semi-norm $|v|_{\sigma,2,\mathbb{R}^n}$ is finite if and only if the integral $\int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi$ is finite. But this is a consequence of (4.7). So let us show that (4.7) holds. To this end, we follow the reasoning of Goulaouic [7, p. 101].

For any $y \in \mathbb{R}^n$, the Fourier transform of the translated function $x \mapsto v(x+y)$ is the function $\xi \mapsto e^{iy \cdot \xi} \hat{v}(\xi)$. Hence, by Parseval's identity, we have

$$\int_{\mathbb{R}^n} |v(x+y) - v(x)|^2 dx = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 |e^{iy \cdot \xi} - 1|^2 d\xi.$$

Then, by Fubini's Theorem and Lemma 4.1, we finally deduce that

$$\begin{aligned} |v|_{\sigma,2,\mathbb{R}^n}^2 &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x+y) - v(x)|^2}{|y|^{n+2\sigma}} dx dy \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 \left(\int_{\mathbb{R}^n} \frac{|e^{iy \cdot \xi} - 1|^2}{|y|^{n+2\sigma}} dy \right) d\xi \\ &= (2\pi)^{-n} G_{\sigma,n} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi, \end{aligned}$$

which yields (4.7) and completes the proof. \square

Lemma 4.4. *Let $\sigma_0 \in (0, 1)$. Then, for any $v \in W^{\sigma_0,2}(\mathbb{R}^n)$,*

$$\lim_{\sigma \rightarrow 0^+} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi = (2\pi)^n |v|_{0,2,\mathbb{R}^n}^2.$$

Proof. Let $v \in W^{\sigma_0,2}(\mathbb{R}^n)$. For any $\sigma \in (0, \sigma_0]$, let us consider the integral $I_\sigma = \int_{\mathbb{R}^n} g_\sigma(\xi) d\xi$, where $g_\sigma(\xi) = (1 - |\xi|^{2\sigma}) |\hat{v}(\xi)|^2$. This integral is well defined: since $v \in W^{\sigma_0,2}(\mathbb{R}^n)$, v also belongs to $L^2(\mathbb{R}^n)$ and $W^{\sigma,2}(\mathbb{R}^n)$, so $\hat{v} \in L^2(\mathbb{R}^n)$ and, by Proposition 4.3, $v \in \widetilde{H}^\sigma(\mathbb{R}^n)$.

Let r and R be numbers such that $0 < r \leq 1 < R$. We set

$$I_\sigma = \int_{|\xi| \leq r} g_\sigma(\xi) d\xi + \int_{r < |\xi| < R} g_\sigma(\xi) d\xi + \int_{|\xi| \geq R} g_\sigma(\xi) d\xi = J_1 + J_2 + J_3.$$

Let $\varepsilon > 0$ be given. Let us show that we can choose r , R and $\sigma \in (0, \sigma_0)$ such that $|I_\sigma| < \varepsilon$. We have

$$|J_1| \leq \int_{|\xi| \leq r} |\hat{v}(\xi)|^2 d\xi.$$

Clearly, $|J_1| \leq \varepsilon/3$ for r small enough, since $\hat{v} \in L^2(\mathbb{R}^n)$. Moreover,

$$|J_3| \leq \int_{|\xi| \geq R} |\hat{v}(\xi)|^2 d\xi + \int_{|\xi| \geq R} |\xi|^{2\sigma_0} |\hat{v}(\xi)|^2 d\xi,$$

and the two terms on the right member are arbitrarily small when R is large enough, the first, because $\hat{v} \in L^2(\mathbb{R}^n)$ and the second, because, by Proposition 4.3, $v \in \widetilde{H}^{\sigma_0}(\mathbb{R}^n)$. So, $|J_3| < \varepsilon/3$ for R sufficiently large. Once r and R chosen, it suffices to take σ small enough to achieve $|J_2| < \varepsilon/3$.

The preceding reasoning implies that

$$\lim_{\sigma \rightarrow 0^+} \int_{\mathbb{R}^n} g_\sigma(\xi) d\xi = 0.$$

Consequently, taking Plancherel's Theorem into account, we conclude that

$$\lim_{\sigma \rightarrow 0^+} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 d\xi = (2\pi)^n |v|_{0,2,\mathbb{R}^n}^2. \quad \square$$

Lemma 4.5. *For any $v \in W^{1,2}(\mathbb{R}^n)$,*

$$\lim_{\sigma \rightarrow 1^-} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi = (2\pi)^n |\nabla v|_{0,2,\mathbb{R}^n}^2.$$

Proof. Let $v \in W^{1,2}(\mathbb{R}^n)$. For any $\sigma \in (0, 1)$, we now consider the integral $I_\sigma = \int_{\mathbb{R}^n} g_\sigma(\xi) d\xi$, with $g_\sigma(\xi) = (|\xi|^2 - |\xi|^{2\sigma}) |\hat{v}(\xi)|^2$. It is clear that $|I_\sigma| < \infty$: on the one hand, the embedding $W^{1,2}(\mathbb{R}^n) \hookrightarrow W^{\sigma,2}(\mathbb{R}^n)$ and Proposition 4.3 imply that $v \in \widetilde{H}^\sigma(\mathbb{R}^n)$; on the other hand, since $v \in W^{1,2}(\mathbb{R}^n)$,

$$\begin{aligned} (4.8) \quad \int_{\mathbb{R}^n} |\xi|^2 |\hat{v}(\xi)|^2 d\xi &= \sum_{|\beta|=1} \int_{\mathbb{R}^n} \xi^{2\beta} |\hat{v}(\xi)|^2 d\xi \\ &= \sum_{|\beta|=1} \int_{\mathbb{R}^n} |i \xi^\beta \hat{v}(\xi)|^2 d\xi = \sum_{|\beta|=1} \int_{\mathbb{R}^n} |\widehat{\partial^\beta v}(\xi)|^2 d\xi \\ &= \sum_{|\beta|=1} (2\pi)^n \int_{\mathbb{R}^n} |\partial^\beta v(x)|^2 dx = (2\pi)^n |\nabla v|_{0,2,\mathbb{R}^n}^2, \end{aligned}$$

which is finite.

As in the proof of Lemma 4.4, we set

$$I_\sigma = \int_{|\xi| \leq r} g_\sigma(\xi) d\xi + \int_{r < |\xi| < R} g_\sigma(\xi) d\xi + \int_{|\xi| \geq R} g_\sigma(\xi) d\xi = J_1 + J_2 + J_3,$$

with r and R such that $0 < r \leq 1 < R$. Let $\varepsilon > 0$ be given. Clearly, we have

$$|J_1| \leq 2 \int_{|\xi| \leq r} |\hat{v}(\xi)|^2 d\xi \quad \text{and} \quad |J_3| \leq 2 \int_{|\xi| \geq R} |\xi|^2 |\hat{v}(\xi)|^2 d\xi.$$

Then, the assumption $v \in W^{1,2}(\mathbb{R}^n)$ implies that r and R can be chosen in such a way that $|J_1|$ and $|J_3|$ be $\leq \varepsilon/3$. We have just to take σ sufficiently close to 1 to achieve $|J_2| < \varepsilon/3$. Consequently,

$$\lim_{\sigma \rightarrow 1^+} \int_{\mathbb{R}^n} g_\sigma(\xi) d\xi = 0.$$

From this relation and (4.8), we finally derive that

$$\lim_{\sigma \rightarrow 1^-} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\xi|^2 |\hat{v}(\xi)|^2 d\xi = (2\pi)^n |\nabla v|_{0,2,\mathbb{R}^n}^2. \quad \square$$

We are now ready to prove the main result in this section, which establishes Theorems 3.3 and 3.4 in the particular case $p = 2$. The reader may want to check that the constants on the right-hand side of (3.6) and (3.7) are equal, for $p = 2$, to those in (4.10) and (4.9), respectively.

Theorem 4.6. *Let $l \in \mathbb{N}$.*

(i) *Let $\sigma_0 \in (0, 1)$. Then, for any $v \in W^{l+\sigma_0, 2}(\mathbb{R}^n)$,*

$$(4.9) \quad \lim_{\sigma \rightarrow 0^+} \sigma |v|_{l+\sigma, 2, \mathbb{R}^n}^2 = \frac{2\pi^{n/2}}{\Gamma(n/2)} |v|_{l, 2, \mathbb{R}^n}^2.$$

(ii) *For any $v \in W^{l+1, 2}(\mathbb{R}^n)$,*

$$(4.10) \quad \lim_{\sigma \rightarrow 1^-} (1 - \sigma) |v|_{l+\sigma, 2, \mathbb{R}^n}^2 = \frac{\pi^{n/2}}{n\Gamma(n/2)} |\nabla v|_{l, 2, \mathbb{R}^n}^2.$$

Proof. Let us first assume that $l = 0$. It readily follows from (4.3) and the continuity and properties of the Γ function that

$$\lim_{\sigma \rightarrow 0^+} \sigma G_{\sigma, n} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{and} \quad \lim_{\sigma \rightarrow 1^-} (1 - \sigma) G_{\sigma, n} = \frac{\pi^{n/2}}{n\Gamma(n/2)}.$$

Consequently, by Proposition 4.3 and Lemma 4.4, we have

$$\lim_{\sigma \rightarrow 0^+} \sigma |v|_{\sigma, 2, \mathbb{R}^n}^2 = \lim_{\sigma \rightarrow 0^+} \sigma (2\pi)^{-n} G_{\sigma, n} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi = \frac{2\pi^{n/2}}{\Gamma(n/2)} |v|_{0, 2, \mathbb{R}^n}^2.$$

Likewise, by Proposition 4.3 and Lemma 4.5,

$$\begin{aligned} & \lim_{\sigma \rightarrow 1^-} (1 - \sigma) |v|_{\sigma, 2, \mathbb{R}^n}^2 \\ &= \lim_{\sigma \rightarrow 1^-} (1 - \sigma) (2\pi)^{-n} G_{\sigma, n} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 d\xi = \frac{\pi^{n/2}}{n\Gamma(n/2)} |\nabla v|_{0, 2, \mathbb{R}^n}^2. \end{aligned}$$

The reasoning for $l \geq 1$ follows the same pattern already shown in Theorems 3.3 and 3.4. \square

In the proof of Theorem 4.6 and the preceding lemmas, Proposition 4.3 plays a fundamental role. This result can be extended to characterize the space $W^{r, 2}(\mathbb{R}^n)$ for any $r \geq 0$. Although it is not required here, we include such an extension in this section for the sake of completeness.

Theorem 4.7. *Let $r \in [0, \infty)$. Then*

$$(4.11) \quad W^{r, 2}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n),$$

where $\widetilde{H}^r(\mathbb{R}^n)$ is given by (4.6) with r instead of σ . Moreover, for any $m \in \mathbb{N}$ and $s \geq 0$ such that $r = m + s$,

$$(4.12) \quad W^{r, 2}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap X^{m, s},$$

where $X^{m, s} = \{v \in \mathcal{D}'(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}^n, |\alpha| = m, \partial^\alpha v \in \widetilde{H}^s(\mathbb{R}^n)\}$, $\mathcal{D}'(\mathbb{R}^n)$ being the space of distributions on \mathbb{R}^n .

Proof. We put $r = l + \sigma$, with $l = \lfloor r \rfloor$ and $\sigma \in [0, 1)$. Let $m \in \mathbb{N}$ and $s \geq 0$ such that $r = m + s$. We remark that $m \leq l$.

Since $L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$, it is clear that

$$(4.13) \quad L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |\xi|^{2r} |\widehat{v}(\xi)|^2 d\xi < \infty \right\}$$

and

$$(4.14) \quad L^2(\mathbb{R}^n) \cap X^{m,s} = \left\{ v \in L^2(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}^n, |\alpha| = m, \right. \\ \left. \widehat{\partial^\alpha v} \in L^1_{\text{loc}}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{\partial^\alpha v}(\xi)|^2 d\xi < \infty \right\}.$$

We divide the proof into several steps: Steps 1 and 2 prove (4.11), whereas Steps 3 and 4 establish (4.12).

STEP 1: $W^{r,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n)$.

Let $v \in W^{r,2}(\mathbb{R}^n)$. By (4.13), we have just to show that $\int_{\mathbb{R}^n} |\xi|^{2r} |\widehat{v}(\xi)|^2 d\xi$ is finite. Let us first consider the case $\sigma \in (0, 1)$. Every l -th derivative $\partial^\alpha v$ belongs to $W^{\sigma,2}(\mathbb{R}^n)$. By Proposition 4.3, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |\xi|^{2r} |\widehat{v}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\xi|^{2l} |\widehat{v}(\xi)|^2 d\xi = \sum_{|\alpha|=l} \binom{l}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2\sigma} \xi^{2\alpha} |\widehat{v}(\xi)|^2 d\xi \\ &= \sum_{|\alpha|=l} \binom{l}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\widehat{\partial^\alpha v}(\xi)|^2 d\xi = \sum_{|\alpha|=l} \binom{l}{\alpha} (2\pi)^n G_{\sigma,n}^{-1} |\partial^\alpha v|_{\sigma,2,\mathbb{R}^n}^2 \\ &\leq M(2\pi)^n G_{\sigma,n}^{-1} \sum_{|\alpha|=l} |\partial^\alpha v|_{\sigma,2,\mathbb{R}^n}^2 = M(2\pi)^n G_{\sigma,n}^{-1} |v|_{r,2,\mathbb{R}^n}^2 < \infty, \end{aligned}$$

with $M = \max \left\{ \binom{l}{\alpha} \mid \alpha \in \mathbb{N}^n, |\alpha| = l \right\}$. If $\sigma = 0$, the above reasoning is still valid, taking $G_{\sigma,n} = 1$ and using Plancherel's Theorem instead of Proposition 4.3.

STEP 2: $L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n) \subset W^{r,2}(\mathbb{R}^n)$.

Let $v \in L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n)$. For any $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq l$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2r} |\widehat{v}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\xi|^{2(r-|\alpha|)} |\xi|^{2|\alpha|} |\widehat{v}(\xi)|^2 d\xi \\ &= \sum_{|\beta|=|\alpha|} \binom{|\alpha|}{\beta} \int_{\mathbb{R}^n} |\xi|^{2(r-|\alpha|)} \xi^{2\beta} |\widehat{v}(\xi)|^2 d\xi. \end{aligned}$$

Consequently, $\widehat{\partial^\alpha v} = i^{|\alpha|} \xi^\alpha \widehat{v}$ belongs to $L^2(\mathbb{R}^n)$, since

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi^\alpha \widehat{v}(\xi)|^2 d\xi &= \int_{|\xi|<1} \xi^{2\alpha} |\widehat{v}(\xi)|^2 d\xi + \int_{|\xi|\geq 1} \xi^{2\alpha} |\widehat{v}(\xi)|^2 d\xi \\ &\leq \int_{|\xi|<1} |\widehat{v}(\xi)|^2 d\xi + \binom{|\alpha|}{\alpha} \int_{|\xi|\geq 1} |\xi|^{2(r-|\alpha|)} \xi^{2\alpha} |\widehat{v}(\xi)|^2 d\xi \\ &\leq \int_{\mathbb{R}^n} |\widehat{v}(\xi)|^2 d\xi + \int_{\mathbb{R}^n} |\xi|^{2r} |\widehat{v}(\xi)|^2 d\xi < \infty. \end{aligned}$$

We deduce from Plancherel's Theorem that $v \in W^{l,2}(\mathbb{R}^n)$. If $\sigma \in (0, 1)$, we still have to see that $|v|_{r,2,\mathbb{R}^n}$ is finite. But a reasoning analogous to that in Step 1 shows, as desired, that

$$|v|_{r,2,\mathbb{R}^n}^2 \leq (2\pi)^{-n} G_{\sigma,n} \int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 d\xi < \infty.$$

STEP 3: $L^2(\mathbb{R}^n) \cap X^{m,s} \subset W^{r,2}(\mathbb{R}^n)$.

Let $v \in L^2(\mathbb{R}^n) \cap X^{m,s}$. Then, taking (4.14) into account, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\xi|^{2s} |\xi|^{2m} |\hat{v}(\xi)|^2 d\xi \\ &= \sum_{|\alpha|=m} \binom{m}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2s} \xi^{2\alpha} |\hat{v}(\xi)|^2 d\xi \\ &= \sum_{|\alpha|=m} \binom{m}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{\partial^\alpha v}(\xi)|^2 d\xi < \infty. \end{aligned}$$

Thus, it follows from (4.11) and (4.13) that $v \in W^{r,2}(\mathbb{R}^n)$.

STEP 4: $W^{r,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap X^{m,s}$.

Let $v \in W^{r,2}(\mathbb{R}^n)$. Using (4.11), the reasoning in Step 2 shows that, for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m$, $\widehat{\partial^\alpha v}$ belongs to $L^2(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$ and

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{\partial^\alpha v}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\xi|^{2s} \xi^{2\alpha} |\hat{v}(\xi)|^2 d\xi \\ &\leq \sum_{|\beta|=m} \binom{m}{\beta} \int_{\mathbb{R}^n} |\xi|^{2s} \xi^{2\beta} |\hat{v}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 d\xi < \infty. \end{aligned}$$

We conclude that, by (4.14), $v \in L^2(\mathbb{R}^n) \cap X^{m,s}$. □

Remark 4.8. Let $r > 0$. Theorem 4.7 allows us to endow $W^{r,2}(\mathbb{R}^n)$ with semi-norms defined in $\widetilde{H}^r(\mathbb{R}^n)$ or $X^{m,s}$. For example, the mapping

$$|\cdot|_{0,r} : v \mapsto \left(\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 d\xi \right)^{1/2}$$

is a semi-norm in $\widetilde{H}^r(\mathbb{R}^n)$ (in fact, a hilbertian norm if $r < n/2$; cf. [1]), so it is in $W^{r,2}(\mathbb{R}^n)$. It follows from steps 1 and 2 in the proof of Theorem 4.7 that $|\cdot|_{0,r}$ and $|\cdot|_{r,2,\mathbb{R}^n}$ are equivalent semi-norms. The equivalence constants depend on σ , since they contain $G_{\sigma,n}$. In fact, taking into account (4.3) and the continuity of the Gamma function, it is readily seen that, given $l \in \mathbb{N}$, there exist constants C_1 and C_2 , depending on n and l , such that, for all $\sigma \in (0, 1)$ and $v \in W^{l+\sigma,2}(\mathbb{R}^n)$,

$$C_1 |v|_{0,l+\sigma} \leq (2\sigma(1-\sigma))^{1/2} |v|_{l+\sigma,2,\mathbb{R}^n} \leq C_2 |v|_{0,l+\sigma}. \quad \square$$

Acknowledgements. This work has been supported by the Ministerio de Ciencia e Innovación (Spain), through the Research Project MTM2009-07315.

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